Quasilinear Functional Differential Equations

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Abstract: In this paper we consider a class of quasilinear functional differential equations in a reflexive Banach space. We apply the method of semidiscretization in time also known as method of lines for proving the existence, uniqueness and continuous dependence on the initial data of strong solution.
Keywords: Quasilinear functional differential equation, Method of lines, Strong solution.

Mathematics Subject Classification (2000): 34G20, 47D06, 47H06, 35B30, 35L55

1. INTRODUCTION

Let X and Y be two real Banach spaces such that the embedding $Y \hookrightarrow X$ is dense and continuous. Consider the following quasilinear functional evolution equation

$$\begin{cases} u'(t) + A\big(u(t)\big)u(t) = f(t, u(t), u_t), & 0 < t \le T, \\ u_0 = \emptyset, \end{cases} \tag{1.1}$$

where $0 < T < \infty$, $A(u)$ is a linear operator in X for each u in an open subset W of Y, for $u \in C([-\tau, T]; X)$, and $t \in [0, T]$ the function $u_t \in C([-\tau, 0]; X)$ given by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-\tau, 0]$, $\emptyset \in C([-\tau, 0]; X)$, $\tau > 0$.

By the strong solution to (1.1) on $[-\tau, T']$, $0 < T' \le T$, we mean a continuous function u from $[-\tau, T']$ into X such that u is absolutely continuous on $[-\tau, T']$, $u(t) \in W$ for almost every $t \in [-\tau, T']$ and satisfies (1.1) almost everywhere on $[-\tau, T']$.

Quasilinear evolution equations forms a very important class of evolution equations as many time dependent phenomena in physics, chemistry and biology can be represented by such evolution equations. For more details on the theory and applications of quasilinear evolution equations we refer to [3], [8], [13].

We use the ideas and techniques of Zeidler [13] and the method of semidiscretization in time to establish existence, uniqueness and continuous dependence on initial data of strong solutions to

(1.1) on $[-\tau, T']$ for some $0 < T' \leq T$. Crandall and Souganidis [4] have proved the existence, uniqueness and continuous dependence of a continuously differentiable solution to the quasilinear evolution equation

$$\frac{du(t)}{dt} + A\big(u(t)\big)u(t) = 0, \qquad 0 < t \leq T, \quad u(0) = u_0 \tag{1.2}$$

T. Kato [5] has proved two general theorems on the nonhomogenous quasilinear evolution equation

$$\frac{du(t)}{dt} + A\big(t, u(t)\big)u(t) = f\big(t, u(t)\big), \qquad 0 < t \leq T, \quad u(0) = u_0 \tag{1.3}$$

one on the existence and uniqueness, and the other on the continuous dependence of a solution on the initial data. Also, he has shown that these theorems are applicable to the different kinds of quasilinear differential equations such as Korteweg-de Vries equation,Navier-stokes equation and Euler equation, equation for compressible fluids etc. The crucial assumptions in the work of Crandall and Souganidis [4], T. Kato [5] and references cited therein is that there exists an open subset W of Y such that for each $w \in W$, $A(w)$ generates a C_0- semigroup in X, $A(.)$ is locally Lipschitz continuous on W from X into X, f, defined from W into$Y$, is bounded and globally Lipschitz continuous from Y into Y, and there exists an isometric isomorphism $S: Y \to X$ such that

$$SA(w)S^{-1} = A(w) + B(w) \tag{1.4}$$

where $B(w)$ is in the space $B(X)$ of all bounded linear operators from X into X.

Abbas and Bahuguna [1] have establish the existence and uniqueness of a classical solution to the following quasilinear functional differential equation considered in Banach space

$$\frac{du(t)}{dt} + A\big(t, u(t)\big)u(t) = F(t, u_t), \qquad 0 \leq t \leq T,$$

$$u(0) = \emptyset \text{ on } [-\tau, 0] \tag{1.5}$$

by using the theory of C_0- semigroup of bounded linear operator under the assumption that for $u \in C_T, F\big(; u(.)\big): [0, T] \to X$ is bounded L^1 function. Further they assume that there is a subset B of X such that for $(t, u) \in [0, T] \times C_T$ with $u(t) \in B$ for $t \in [0, T], A\big(t, u(t)\big)$ is linear operator in X. Also $\emptyset \in C_0$ is Lipschitz continuous with Lipschitz constant $L_\emptyset$.

The method of semidiscretization in time is developed and applied to linear as well as nonlinear evolution equations by Rektorys [9] , Kartsatos and Zigler [10] , Necas [11] , Bahuguna and Raghavendra [2] and others. This method consists in replacing the time derivatives in an evolution equation by the corresponding difference quotients giving rise to a system of time-independent operator equations. With the help of the theory of semigroups and the theory of monotone operators, these systems are guaranteed to have unique solutions. An approximate solution to the evolution equation is defined in terms of the solutions of these time-independent systems. After proving a priori estimates for the approximate solution, the convergence of the approximate solution to the unique solution of the evolution equation is established. In these works either global Lipschitz conditions or local Lipschitz condition with some growth conditions on nonlinear forcing terms have been assumed.

In the present work we assume only local Lipschitz conditions on the nonlinear map f. We first prove that the discrete points lie in a ball in X of fixed radius R where R is independent of the discretization parameters. Then using the local Lipschitz continuity, we establish a priori estimates on the difference quotients. With the help of these a priori estimates we prove the convergence of a sequence of approximate solutions defined in terms of the discrete points to a unique solution of the problem.

2. PRELIMINARIES

Consider X and Y be as in the previous section 1. Let $\|x\|_z$ denote the norm of an element x belonging to a Banach space Z. For $r > 0$, let $B_Z(x,r)$ denote the open ball $\{z \in Z: \|z-x\|_Z < r\}$ of radius r and let $\bar{B}_Z(x,r)$ be its closure. For an interval J of real numbers, we denote by $C(J;Z)$, WC$(J;Z)$, Lip$(J;Z)$, and ABS$(J;Z)$ the spaces of all continuous, weakly continuous, Lipschitz continuous, and absolutely continuous functions from J into Z, respectively.

For a real β, $N(Z,\beta)$ represents the set of all densely defined linear operators L in Z such that if $\lambda > 0$ and $\lambda\beta < 1$, then $(I+\lambda L)$ is one-to-one with a bounded inverse defined everywhere on Z and

$$\|(I+\lambda L)^{-1}\|_{B(Z)} \leq (1+\lambda\beta)^{-1}, \tag{2.1}$$

where I is the identity operator on Z. The Hille-Yosida (cf.Pazy [12]) theorem states that $L \in N(Z,\beta)$ if and only if $-L$ is the infinitesimal generator of a strongly continuous semigroup $e^{-tL}, t \geq 0,$ on Z satisfying $\|e^{-tL}\|_{B(Z)} \leq e^{\beta t}, t \geq 0$. A linear operator L on $D(L) \subseteq Z$ into Z is said to be accretive in Z if for every $u \in D(L)$,

$$\langle Lu, u^* \rangle \geq 0 \text{ for some } u^* \in F(u), \tag{2.2}$$

where $\langle z, z^* \rangle$ is the value of $z^* \in Z^*$ at $z \in Z$ and $F: Z \to 2^{Z^*}$ is the duality map given by

$$F(z) = \{z^* \in Z^*: \langle z, z^* \rangle = \|z\|_Z^2 = \|z^*\|_{Z^*}^2\}. \tag{2.3}$$

Here 2^{Z^*} denotes the power set of Z^*. If $L \in N(Z,\beta)$ then $(L+\beta I)$ is m-accretive in Z, i.e.,$(L+\beta I)$ is accretive and the range $R(L+\lambda I) = Z$ for some $\lambda > \beta$. If Z^* is uniformly convex, then F is single valued and uniformly continuous on bounded subsets of Z.

(H0) We assume, in addition, that the embedding $Y \hookrightarrow X$ is compact and the dual X^* is uniformly convex. Further, we make the following hypotheses.

(H1) There exist an open subset W of Y and $\beta \geq 0$ such that $u_0 \in W$ and

$$A: W \to N(X,\beta); \tag{2.4}$$

(H2) There exist positive constants μ_A and γ_A such that for all $w, w_1, w_2 \in W$ and $v \in Y$,

$$\begin{gathered} Y \subseteq D\big(A(w)\big), \quad \big\|\big(A(w_1) - A(w_2)\big)v\big\|_X \leq \mu_A \|w_1 - w_2\|_X \|v\|_Y, \\ \|A(w)\,v\|_X \leq \gamma_A \|v\|_Y; \end{gathered} \tag{2.5}$$

(H3) There exist a linear isometric isomorphism $S: Y \to X$, a map $P: W \to B(X)$, and positive constants μ_P and γ_P such that for all $w, w_1, w_2 \in W$,

$$\begin{gathered} SA(w) = A(w)S + P(w)S, \quad \|P(w)\|_X \leq \gamma_P, \\ \|P(w_1) - P(w_2)\|_X \leq \mu_P \|w_1 - w_2\|_Y; \end{gathered} \tag{2.6}$$

(H4) The nonlinear map $f: [0,T] \times Y \times C([-\tau;0];Y) \to Y$ is a bounded function:

$$\|f(t,u,v)\|_Y \leq \lambda_f(r) \tag{2.7}$$

for all $(t,u,v) \in [0,T] \times Y \times C([-\tau;0];Y)$ with $\|u\|_Y + \|v\|_{C([-\tau,0];Y)} \leq r$ where $\lambda_f(r)$ is a nonnegative nondecreasing function. Also, this map satisfies the Lipschitz condition

$$\|f(t_1,u_1,v_1) - f(t_2,u_2,v_2)\|_X \leq \mu_f(r)\,[|\psi(t_1) - \psi(t_2)| + \|u_1 - u_2\|_X + \|v_1 - v_2\|_{C([-\tau,0];X)}]$$

for all $t_1, t_2 \in [0,T]$ and all $u_i \in \overline{B}_X(\phi(0), r)$, and $v_i \in \overline{B}_{C([-\tau,0];X)}(\phi, r)$, $i = 1,2$, where ψ is a real-valued continuous function of bounded variation on $[0,T]$ and $\mu_f(r)$ is a nonnegative nondecreasing function.

Let $R > 0$ be such that $W_R = \overline{B}_Y(u_0, R) \subseteq W$. We set

$$R_0 = \frac{R}{2}(1 + e^{2\theta T})^{-1} \tag{2.8}$$

$$R_1 = \max\,\{4R + \|u_0\|_Y\} \tag{2.9}$$

$$M = \lambda_f(4R + \|u_0\| \tag{2.10}$$

where $\theta = \beta + \|P\|_{B(X)}$ and $V(\psi)$ is the total variation of ψ on$[0,T]$.

Let $z_0 \in Y$ and $T_0, 0 < T_0 \le T$ be such that

$$\|Su_0 - z_0\|_X \le R_0 \tag{2.11}$$

$$T_0[\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M] \le R_0 \tag{2.12}$$

We note that (2.11) and (2.12) imply that

$$(1 + e^{2\theta T})[\|Su_0 - z_0\| + T_0\{\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M\}] \le 2R/3 \tag{2.13}$$

We have the following main result for the existence, uniqueness and continuous dependence on the initial data of the strong solutions to (1.1).

Theorem 2.1 Let (H0)-(H4) hold and let ϕ be Lipschitz continuous on$[-\tau, 0]$. Than there exists a unique strong solution u to (1.1) on $[-\tau, T_0]$ such that $u \in Lip([0,T_0];X)$. Furthermore, if $\tilde{u}(0) \in \overline{B_Y}(u_0), R_0)$ then there exists a strong solution $\tilde{u}$ to (1.1) on $[0, T_0]$ with the initial point u_0 replaced by $\breve{u}(0)$ and

$$\|u(t) - \tilde{u}(t)\|_X \le C\|u(0) - \tilde{u}(0)\|_X, \quad t \in [0, T_0]$$

where C is a positive constant depending only on T_0 .

3. BASIC LEMMAS

We shall state and prove several lemmas required to prove Theorem 2.1. A proof of theorem 2.1 will be given in the next section.

Let $w_0 = Su_0$. Let $h = {}^{T_0}/_n$ for all positive integers $n \ge N$ where N is a positive integer such that $\theta\left({}^{T_0}/_N\right) < {}^1/_2$. For $n \ge N$, we set $u_0^n = \phi(0)$, $\tilde{u}_0^n = \tilde{u}_0$ and $t_j^n = jh$

For $j = 1,2,3, \dots, n$.We consider the scheme

$$\delta u_j^n + A(u_{j-1}^n)u_j^n = f\left(t_j^n, u_{j-1}^n, (\tilde{u}_{j-1}^n)_{t_j^n}\right), \quad j = 1,2, \dots, n, \tag{3.1}$$

where, for $j = 1,2, \dots, n, n \ge N,$

$$\delta u_j^n = \frac{u_j^n - u_{j-1}^n}{h}, \tag{3.2}$$

$$\tilde{u}_j^n(t) = \begin{cases} \phi, & t \in [-\tau, 0] \\ u_{i-1}^n + (1/h)(t - t_{i-1}^n)(u_i^n - u_{i-1}^n), & t \in [t_{i-1}^n, t_i^n], i = 1,2, \dots, j, \\ u_j^n, & t \in [t_j^n, T_0] \end{cases} \tag{3.3}$$

The following result establishes the fact that $u_j^n \in W_R$, $j = 1,2,\dots,n, n \geq N$.

Lemma 3.1 For each $n \geq N$, there exist unique $u_j^n \in W_R, j = 1,2,\dots,n$ satisfying (3.1)

Proof: It follows (cf. Lemma 2 in [4]) that there exists a unique $u_1^n \in Y$ such that

$$u_1^n + hA(u_0)u_1^n = u_0 + hf(t_1^n, u_0, (\tilde{u}_0)_{t_1^n}). \tag{3.4}$$

Applying S on both sides in (3.4) using (H3) and putting $w_1^n = Su_1^n$, we have

$$(w_1^n - z_0) + hA(u_0)(w_1^n - z_0) + hP(u_0)(w_1^n - z_0) = (w_0 - z_0) - hA(u_0)z_0 + hP(u_0)z_0 + hSf(t_1^n, u_0, (\tilde{u}_0)_{t_1^n}).$$

The estimates in Lemma 2 in [4] imply that

$$\|w_1^n - z_0\|_X \leq (1 - h\theta)^{-1}[\|w_0 - z_0\| + h(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)].$$

Since $h\theta < 1/2$, we have

$$\|w_1^n - z_0\|_X \leq e^{2\theta h}[\|w_0 - z_0\| + h(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)].$$

Therefore

$$\|w_1^n - w_0\|_X \leq \left(1 + he^{2\theta h}\right)[\|w_0 - z_0\| + h(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)] \leq R$$

In view of the estimate (2.7). Hence $u_1^n \in W_R$. Now suppose that $u_i^n \in W_R$ for $i = 1,2,\dots,j-1$. Again, Lemma 2 in [4] implies that for $2 \leq j \leq n$ there exist unique $u_j^n \in Y$ such that

$$u_j^n + hA\left(u_{j-1}^n\right)u_j^n = u_{j-1}^n + hf(t_j^n, u_{j-1}^n, (\tilde{u}_{j-1}^n)_{t_j^n}. \tag{3.5}$$

Proceeding as before and putting$w_j^n = Su_j^n$, we get the estimate

$$\left\|w_j^n - z_0\right\|_X \leq e^{2\theta h}[\left\|w_{j-1} - z_0\right\|_X + h(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)].$$

Reiterating the above inequality, we get

$$\left\|w_j^n - z_0\right\|_X \leq e^{2\theta jh}[\|w_0 - z_0\|_X + jh(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)].$$

Using the fact that $jh \leq T_0$, we arrive at

$$\left\|w_j^n - w_0\right\|_X \leq (1 + e^{2\theta T})[\|w_0 - z_0\|_X + T_0(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)] \leq R.$$

The above estimate and the equation (3.4) and (3.5) gives the required result. This completes the proof.

Lemma 3.2 There exists a positive constant C, independent of j, h and n such that

$\|\delta u_j\|_X \leq C, \quad j = 1,2,3, \dots, n; n \geq N.$

Proof: Putting $j = 1$ in (3.1) we get

$\delta u_1^n + hA(u_0)(\delta u_1^n) = -A(u_0)u_0 - f(t_1^n, u_0, (\tilde{u}_0)_{t_1^n}).$

Using Lemma 2 of [4], we have

$$\|\delta u_1^n\|_X \leq e^{2\theta T}[\gamma_A \|u_0\|_Y + M] = C_0. \tag{3.6}$$

From (3.1) for$2 \leq j \leq n$, we have

$$\delta u_j^n + hA(u_{j-1}^n)(\delta u_j^n) = \delta u_{j-1}^n - \left(A(u_{j-1}^n) - A(u_{j-2}^n)\right) u_{j-1}^n + f\left(t_j^n, u_{j-1}^n, (\tilde{u}_{j-1}^n)_{t_j^n}\right) - f\left(t_{j-1}^n, u_{j-2}^n, (\tilde{u}_{j-2}^n)_{(t_{j-1}^n)}\right)$$

Using (H2) and Lemma 2 of [4], for $2 \leq j \leq n$, we get

$$\|\delta u_j^n\|_X \leq e^{2\theta h}[(1 + C_1 h)\|\delta u_{j-1}^n\|_X + \left\|f\left(t_j^n, u_{j-1}^n, (\tilde{u}_{j-1}^n)_{t_j^n}\right) - f\left(t_{j-1}^n, u_{j-2}^n, (\tilde{u}_{j-2}^n)_{(t_{j-1}^n)}\right)\right\|_X] \tag{3.7}$$

For some positive constant C_1 independent of j, h and n.Now,

$$\left\|f\left(t_j^n, u_{j-1}^n, (\tilde{u}_{j-1}^n)_{t_j^n}\right) - f\left(t_{j-1}^n, u_{j-2}^n, (\tilde{u}_{j-2}^n)_{(t_{j-1}^n)}\right)\right\|_X \leq \mu_f(R_1)[|\psi(t_j^n) - \psi(t_{j-1}^n)| + h(1 + \max_{1 \leq i \leq j-1}\|\delta u_i^n\|_X)]$$

.

(3.8)

Using (3.8) in (3.7), we obtain

$$\max_{1 \leq i \leq j}\|\delta u_i^n\|_X \leq e^{2\theta h}(1 + C_2 h)[\max_{1 \leq i \leq j-1}\|\delta u_i^n\|_X + C_2|(\phi t_j^n) - \phi(t_{j-1}^n)| + C_2 h], \tag{3.9}$$

where C_2 is another positive constant independent of $, h$ *and* n . Reiterating inequality (3.9) and using (3.6), we get

$$\max_{1 \leq i \leq j}\|\delta u_i^n\|_X \leq e^{2\theta jh}(1 + C_1 h)^j[\|\delta u_1^n\|_X + C_2 V(\phi) + C_2 T_0]$$

where $V(\psi)$ is total variation of ψ. Hence

$$\|\delta u_i^n\|_X \leq e^{2(\theta + C_2)T}[c_0 + c_2 V(\psi) + C_2 T_0] = C$$

This completes the proof.

Now we define a sequence of functions $\{U^n\}$ from J_0 into Y defined by

$$U^n(t) = u_{j-1}^n + (1/h)(t - t_{j-1})(u_j^n - u_{j-1}^n), t \in [t_{j-1}^n, t_j^n], j = 1,2,\dots,n. \tag{3.10}$$

Furthermore, we define a sequence of step functions $\{X^n\}$ from $(-h, T_0]$ into Y given by

$$X^n(t) = \begin{cases} u_0, & t \in (-h, 0] \\ u_j, t \in (t_{j-1}^n, t_j^n], & j = 1,2,\dots,n. \end{cases} \tag{3.11}$$

Remark 3.3 We observe that $X^n(t) \in W_R$ for all $t \in (-h, T_0]$ and $n \geq N$. Also, $X^n(t) - U^n(t) \to 0$ in X uniformly on J_0 as $n \to \infty$ and $\{U^n\}$ are in $Lip(J_0, X)$ with uniform Lipschitz constant C.

For notational convenience, we shall denote by

$$f^n(t) = f(t_j^n, u_{j-1}^n, (\tilde{u}_{j-1}^n)_{t_j^n}, t \in (t_{j-1}^n, t_j^n], j = 1,2,\dots,n. \tag{3.12}$$

We note that

$$\int_0^t A(X^n(s-h)X^n(s)ds = u_0 - U^n(t) + \int_0^t f^n(s)ds. \tag{3.13}$$

And

$$\frac{d}{dt}U^n(t) + A(X^n(t-h))X^n(t) = f^n(t). \tag{3.14}$$

Lemma 3.4 There exist a subsequence $\{U^m\}$ of $\{U^n\}$ and a function u in $Lip(J_0, X)$ such that $U^m \to u$ in $C(J_0, X)$ (with the supremum norm) as $m \to \infty$.

Proof: Since $\{X^n\}$ is uniformly bounded in Y, the compact embedding of Y implies there exist a subsequence $\{X^m\}$ of $\{X^n\}$ and a function $u: J_0 \to X$ such that $X^m(t) \to u(t)$ in X as $m \to \infty$.

The reflexivity of Y implies that $u(t)$ is the weak limit of X^m in Y hence $u(t)$ lies in Y(in fact in W_R since $X^m(t)$ is in W_R). Now, $X^m(t) - U^m(t) \to 0$ in X so $U^m(t) \to u(t)$ as $m \to \infty$. The uniform continuity of $\{U^n\}$ on J_0 implies that $\{X^m\}$ is an equi-continuous family in $C(J_0, X)$ and the strong convergence of $U^m(t)$ to $u(t)$ in X implies that $\{U^m\}$ is relatively compact in X. We use the Ascoli-Arzela theorem to conclude that $U^m \to u$ in $C(J_0, X)$ as $m \to \infty$. Since U^m are in $Lip(J_0, X)$ with uniform Lipschitz constant, $u \in Lip(J_0, X)$. This completes the proof of the lemma.

Poof of the theorem 2.1

First we show that $A(X^m(t-h))X^m(t) \rightharpoonup A(u(t))u(t)$ in X as $m \longrightarrow \infty$, where '$\rightharpoonup$' denotes the weak convergence in X.

$A(X^m(t-h))X^m(t) - A(u(t))u(t) =$

$$\left(A\left(X^m(t-h)\right)-A\left(u(t)\right)\right)X^m(t)+A\left(u(t)\right)\left(X^m(t)-u(t)\right). \qquad (4.1)$$

Now,

$$\left\|\left(A\left(X^m(t-h)\right)-A\left(u(t)\right)\right)X^m\right\|_X \leq \mu_A\left(R+\|u_0\|_Y\right)\|X^m(t-h)-u(t)\|_X \to 0, \qquad (4.2)$$

As $m \to \infty$, since $X^m(t) \to u(t)$ in X uniformly on J_0. Since $A\left(u(t)\right) \in N(X,\beta), \beta I + A(u)$ is m-accretive in X.We use lemma 2.5 of Kato [5] and the fact that

$$\left\|A\left(u(t)\right)\left(X^m(t)-u(t)\right)\right\|_X \leq 2\gamma_A R \qquad (4.3)$$

to assert that $A\left(u(t)\right)X^m(t) \rightharpoonup A\left(u(t)\right)u(t)$ in X and, hence, $A\left(X^m(t-h)\right)X^m(t) \rightharpoonup A\left(u(t)\right)u(t)inX$, as $m \to \infty$.To show that $A\left(u(t)\right)u(t)$ is weakly continuous on J_0, let $\{t_k\} \subset J_0$ be a sequence such that $t_k \to t$, as $k \to \infty$. Then $u(t_k) \to u(t)$ in X as $k \to \infty$ and we may follow the same arguments as above to prove the $A\left(u(t_k)\right)u(t_k) \rightharpoonup A\left(u(t)\right)u(t)$ in X as $k \to \infty$. The Bochner integrability of $A\left(u(t)\right)u(t)$ can be established in the similar way as in lemma 4.6 in Kato [5]. Now from (3.13), for each $x^* \in X^*$ we have

$$< U^m(t), x^* > = < u_0, x^* > + \int_0^t < -A\left(X^m(s-h)\right)X^m(s) + f^m(s), x^* > ds.$$

Letting $m \to \infty$ using bounded convergence theorem and lemma 3.4, we get

$$< u(t), x^* > = < u_0, x^* + \int_0^t < -A(u(s))u(s) + f(s, u(s), u_s), x^* > ds.$$

The continuity of the integrand implies that $< u(t), x^* >$ is continuously differentiable on J_0 and

$$u'(t) + A\left(u(t)\right)u(t) = f(t, u(t), u_t), \text{ a.e. on } J_0. \qquad (4.4)$$

Since $u = \phi$ on$[-\tau, 0]$, u is a strong solution to (1.1).

Now we establish the uniqueness and the continuous dependence on the initial data of a strong solution to (1.1).

Uniqueness: Let v be another strong solution to (1.1) on J_0. Let $U = u - v$. Then for a.e. $t \in J_0$

$$\left(\frac{dU}{dt}(t), F(U(t))\right) + < \left(\beta I + A\left(u(t)\right)\right)U(t), F\left(U(t)\right) > = \beta\|U(t)\|_X^2 + < \left(A\left(u(t)\right) - A\left(v(t)\right)\right)v(t), F\left(U(t)\right) > + < f(t, u(t), u_t) - f(t, v(t), v_t, F\left(U(t)\right) > \qquad (4.5)$$

Using m –accretivity of $\beta I + A(w)$ and the assumptions on $A(w)$ for $w \in W$ and $f(t, u, v)$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|^2 \leq C_R\|U\|^2_{C(J_t, X)}, \qquad (4.6)$$

where $J_t = [0,t]$ and $C_T = \beta + \mu_A R + \mu_f(R)[1 + \mu_G(R)]$.

Integrating the above inequality on $(0,t)$ and taking the supremum we get

$$\frac{1}{2}\|U\|^2_{C(J_t,X)} \leq C_R \int_0^t \|U\|^2_{C(J_t,X)} ds. \tag{4.7}$$

Applying Gronwall's inequality we get $U \equiv 0$ on J_0.

Continuous Dependence: Let $v_0 \in B_Y(u_0, R_0)$ .Then

$$\|Sv_0 - z_0\|_X \leq \|Sv_0 - Su_0\|_X + \|Su_0 - z_0\|_X \leq 2R_0. \tag{4.8}$$

Hence

$$(1 + e^{2\theta T})[\|Sv_0 - z_0\| + T_0\{\gamma_A \|z_0\|_Y + \gamma_P \|z_0\|_X + M\}] \leq 3(1 + e^{2\theta T})R_0 = R. \tag{4.9}$$

We may proceed as before to prove the existence of $v_j^n \in W_R$ satisfying scheme (1.1) with u_j^n and u_0 replaced by v_j^n andv_0, respectively. Convergence of v_j^n to $v(t)$ can be proved in similar manner.

Let $U = u - v$.Then following the steps used to prove the uniqueness, we have for a.e. , $t \in J_0$,

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|^2_X \leq C_R \|U\|^2_{C(J_t,X)}. \tag{4.10}$$

Integrating the above inequality on $(0,t)$ and taking the supremum we get

$$\frac{1}{2}\|U\|^2_{C(J_t,X)} \leq \frac{1}{2}\|U(0)\|^2_X + C_R \int_0^t \|U\|^2_{C(J_t,X)} ds. \tag{4.11}$$

Applying Gronwall's inequality we get

$$\|U\|_{C(J_t,X)} \leq C\|U(0)\|_X, \tag{4.12}$$

where C is a positive constant. This proves the required result. This completes the proof of theorem 2.1.

Raeferences

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